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Nutational Stability and Core Energy of a Quasirigid Gyrostat

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The asymptotic nutational stability of a quasirigid gyrostat is analyzed. The primary purpose of this analysis is to resolve a debate concerning the use of the energy-sink method of analysis for systems containing driven rotors. It is shown that when the work done by the motor torque is not taken into account, the analysis leads to a contradiction even when the total energy is dissipative. A proper application of Landon's original idea yields a relationship between the time rate of change of Hubert's "core energy" and the energy dissipation rate of the damping mechanisms in the spacecraft. The analysis shows that the core energy might increase during a rotor despin condition; hence, the minimality of core energy—a previous criterion—is not always guaranteed. A criterion for the design of the damper to insure dissipation of the core energy is presented; this condition is always satisfied for the case of a constant relative rotor spin speed that facilitates a "closed-form" solution to the nutation angle time history of an axisymmetric gyrostat. The stability condition resulting from this analysis is consistent with the Landon-Iorillo stability criterion.

Nomenclature

a_i = dextral orthonormal triad fixed on the platform but aligned along I_i, $i = 1, 2, 3$
b_3 = unit vector along the rotor spin axis and parallel to a_3
b = distance between the damper particle and the spin axis (see Fig. 3)
c = linear viscous damper constant
E = total energy of the gyrostat (considered as a two-body system)
E_C = core energy
E_D = energy dissipation due to damping mechanisms
$\dot{E}_P$ = energy dissipation rate in the platform
$\dot{E}_R$ = energy dissipation rate in the rotor
H = (central) angular momentum of the gyrostat
I_i = principal moments of inertia of a gyrostat, $i = 1, 2, 3$
I_s = moment of inertia of an axisymmetric gyrostat about the spin axis
I_t = moment of inertia of an axisymmetric gyrostat about the transverse axis
J = axial moment of inertia of the rotor
k = linear spring constant
m_G = mass of the gyrostat
m_p = mass of the particle in the discrete damper case
Q = quantity defined in Eq. (30)
q = spring deflection of the discrete damper
T_P = net torque on the platform
$T_{P/M}$ = motor torque on the platform
T_R = net axial torque on the rotor
$T_{R/M}$ = motor torque on the rotor
$\dot{W}$ = rate of work done by the motor torque
η = nutation angle
λ_0 = inertial nutation frequency
λ_P = platform nutation frequency
λ_R = rotor nutation frequency
Ω = relative spin rate of the rotor
ω_i = components of the inertial angular velocity of the platform along the principal axes of the gyrostat, $i = 1, 2, 3$

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Introduction

SINCE the pioneering work of Vernon D. Landon, establishing stability criteria for the class of "slightly nonrigid" or quasirigid spacecraft has received widespread attention,[1-5] particularly the heuristic "energy-sink" analysis. However, over the years, fueled by a "lack of rigor" in the analysis, the approach has incited considerable debate,[6-10] sometimes because of the interpretation of the method. A primary source of debate has been the argument that the energy-sink approach is invalid for systems containing driven rotors since the motors could act as energy sources.[6,7] As early as 1964, Landon and Stewart[5] had addressed this problem, and by subtracting the work done by the motor torque from the kinetic energy, they effectively showed that the results continue to hold; however, this concept appears to have been overlooked because of their "restrictive analysis" and the debate continued.

The term energy sink is perhaps misleading since it conjures up the image of total energy dissipation; this is perhaps the source of the misuse. As first noted in Ref. 5, the essence of the energy-sink idea is still valid even in the presence of possible energy sources, as long as the energy source (motor) is properly taken into account. This is an easy task when one part of the two-body system is rigid and axisymmetric; Euler's dynamical equations may be used for the rigid part of the system and axisymmetry eliminates the gyroscopic term. The challenge is in quantifying the energy source when both of the bodies are quasirigid. Inability to do this usually results in a formal development of the stability criteria by assuming that the motor is absent and the bearings are frictionless, or effectively, the motor applying just enough torque to overcome the bearing friction.[2,11] The danger is in using the ensuing stability criterion even when the motor is altering the (mechanical) energy by, for instance, keeping the relative rotor spin rate a constant. The predilection to do this is, perhaps, traceable to the rigid-body solution of an axisymmetric gyrostat[12] where it is possible to keep the rotor relative spin rate a constant without applying a motor torque; however, when the nutation angle *changes*, a nonzero motor torque *must* be applied to maintain a constant relative spin. This fact has been pointed out before in Ref. 10 wherein it is also shown that the energy-sink analysis agrees with the first-order solution for the nutational motion of a spacecraft containing a discrete damper. We wish to emphasize here that the aforementioned special case of a constant relative spin rate is not just of academic interest; rather, it assumes paramount importance in practice because of the ease in maintaining this condition by a simple control law, as in the case of INTELSAT VI.[6]

The problem at hand is the following: spacecraft have been designed on the basis of Landon's stability criterion[1] (as modified by Iorillo[3,4] and hereafter referred to as the Landon-Iorillo stability criterion) and have flown successfully, and the instabilities caused by a rotor energy dissipation have been tested in flight. In particular, the energy-sink predictions for INTELSAT VI have been verified when it satisfies the assumptions of quasirigidity.[6] Yet the formal development of the energy-sink stability criterion for the two-quasirigid-body system does not include the work done by the motor. Apparently, the stability criterion seems to work even in this adverse situation.

The primary contribution of this paper is the "re-establishment" of the energy-sink hypothesis by furthering the ideas of Ref. 5 in showing the validity of the method in the presence of a motor torque. This analysis begins by showing exactly why the conclusions of Ref. 7 contradict the Landon-Iorillo stability criterion. This is followed by the development of a formula that enables one to find the nutation angle time history based on a postulated energy dissipation function because of the nutation dampers. A stability condition derived from this formula agrees with previously known criteria. All of this is made possible by deriving a relationship between Hubert's concept of "core energy"[9] and the damper energy. For a constant relative rotor spin, this expression reduces to the equivalence of the energy dissipation due to the damper and the core energy (dissipation) rates. For an assumed dissipation function, the nutation angle thus predicted agrees with numerical simulations of an "exact" discrete damper model. In contrast, the assumption of total kinetic energy dissipation produces well-known discrepancies.

Background

The main purpose of this section is to show an inconsistency in the purported axiom of the energy-sink analysis that the total energy is always dissipative. To see this, the Landon-Iorillo energy-sink stability criterion may be stated as follows[2-5]:

$$\frac{\dot{E}_R}{\lambda_R} + \frac{\dot{E}_P}{\lambda_P} \leq 0 \tag{1}$$

where the "body-fixed nutation frequencies" are defined according to

$$\lambda_P = \lambda_0 - \omega_3 \tag{2a}$$

$$\lambda_R = \lambda_P - \Omega \tag{2b}$$

and the inertial nutation frequency (for an axisymmetric gyrostat) is given by

$$\lambda_0 = \frac{I_s\omega_3 + J\Omega}{I_t} \tag{2c}$$

The nutation angle is defined in the usual manner (see Fig. 1) according to

$$\eta = \cos^{-1}\left(\frac{b_3 \cdot H}{\|H\|}\right) \tag{3}$$

where $\|H\|$ denotes the magnitude of H. Further, since

$$b_3 \cdot H = J\Omega + I_s\omega_3 \tag{4}$$

the condition

$$\lambda_0 > 0 \tag{5}$$

indicates that the analysis of the nutational motion is restricted to the region

$$0 \leq \eta < \pi/2 \tag{6}$$

Since in general, for a prolate (i.e., $I_s < I_t$) gyrostat, λ_R is negative, we can conclude that energy dissipation in the platform is stabilizing whereas that in the rotor is destabilizing. It goes without saying that stability here refers to nutational stability (i.e., decreasing nutation angle). The point we wish to emphasize here is that according to these equations a *prolate dual spinner is stabilizable*. As pointed out in the introductory section, the analysis and hence the conclusion are valid (within the limits of quasirigidity) when the motor is effectively absent; however, the analysis is invalid when the motor is (effectively) present, but this conclusion based on Eq. (1) *appears* to be valid (from flight test and numerical simulations).

Although the foregoing equations describe the stability criterion, they do not predict the nutational motion quantitatively. To establish this, we can write the nutation angle from Eqs. (3) and (4) as

$$\eta = \cos^{-1}\left[(J\Omega + I_s\omega_3)/H\right] \tag{7}$$

As shown in Ref. 7, the following expression for the angular velocity component ω_3 (sometimes called the "spin component") may be found in terms of the (total) kinetic energy and the angular momentum,

$$\omega_3 = -\frac{J\Omega}{I_s} \pm \sqrt{\left(\frac{J\Omega}{I_s}\right)^2 - \frac{H^2 - 2I_tE + J\Omega^2(I_t - J)}{I_s(I_t - I_s)}} \tag{8}$$

By substituting this into Eq. (7), one arrives at

$$\eta = \cos^{-1}\{\pm\sqrt{\tilde{Q}}\} \tag{9}$$

where

$$\tilde{Q} = \frac{(2E - J\Omega^2)I_s + (J\Omega)^2}{H^2(1 - I_s/I_t)} - \frac{I_s}{I_t - I_s} \tag{10}$$

In general, the sign ambiguity may be resolved as follows: from the initial conditions and Eq. (8), the sign is determined at $t = 0$; thereafter, from continuity considerations the sign is determined for all time. However, from Eqs. (6) and (9), it is clear that the positive sign is consistent. Choosing the upper sign for Eq. (9) can, however, make the nutation angle grow. To see this, we first note that the relative rotor spin is a control variable and one may arbitrarily select it, say $\Omega(t) = \Omega$, a constant. Since H is a constant (torque-free motion), differentiation of Eq. (9) yields

$$\dot{\eta} = \pm\frac{1/\sqrt{\tilde{Q}}}{\sin\eta}\,\frac{(-\dot{E})I_s}{H^2(1 - I_s/I_t)} \tag{11}$$

Thus, according to this equation, energy dissipation in a prolate gyrostat will make the nutation angle grow [for the upper sign, i.e., whenever the condition of Eq. (6) is met]: *a prolate dual spinner is not stabilizable*. Note that this reasoning works both ways; that is, if the lower sign is chosen from stability considerations arising from Eq. (11), then the condition of Eq. (6) will be violated. Clearly this leads to a "flat spin" condition. (Also note that the upper sign is consistent in Ref. 7 and the numerical examples in this paper.)

We thus have the following dilemma: Eq. (1) implies the possibility of a stable prolate gyrostat whereas Eq. (11) contradicts it and vice versa. Thus, the energy-sink theory used heretofore is self-contradictory!

In the following section, an alternative formula for the nutation angle, consistent with the conclusion based on Eq. (1), is presented.

Analysis

In the foregoing section, we have seen that the assumption of total energy dissipation for a gyrostat leads to contradictions. We wish to emphasize that this conclusion is independent of whether the energy actually *decreases or increases*. For example, for a rigid rotor and a quasirigid platform, if the total energy decreases (increases), Eq. (1) predicts stability (instability) whereas Eq. (11) predicts instability (stability). It is important to recognize this point since, in Ref. 7, Eq. (9) has been used to conclude that the energy-sink analysis is inapplicable to systems containing driven rotors because of the possibility of energy addition; on the other hand, from the analysis of the preceding section, the conclusion is different and much stronger: the energy-sink analysis is inherently incorrect *if* the *total energy* is considered to be decreasing *or* increasing. As was first pointed out in Ref. 5, if the energy-sink analysis is to be applied correctly, we *must* take into account the power expended or absorbed by the motor; otherwise, one arrives at an erroneous conclusion. The preceding section best illustrates this for the case when the rotor is maintained at a constant relative speed. The rate of change of the gyrostat energy must, therefore, be written as

$$\dot{E} = \dot{E}_D + \dot{W} \tag{12}$$

Alternatively, this equation may be viewed as the definition for the effective rate of change of energy dissipation due to all damping mechanisms in the system. Thus, a primary goal of the present energy-sink analysis is the determination of $\dot{E}_D$ via Eq. (12).

The main contribution of this paper is a first step in the reconciliation of the energy-sink theory by putting forth the correct expression for the nutation angle time history. In Ref. 9, it was pointed out that the core energy is the correct parameter (it is a decreasing quantity). Here we show that this is not always true, although it does take a special significance for the case of constant relative rotor speed.

Relationship Between Core Energy and Energy Dissipation

Although most of this paper is concerned about the behavior of an axisymmetric quasirigid gyrostat, the discussion in this section applies to a more general gyrostat.

Since the work done by the motor torque is identified to be fundamentally important, the major emphasis in an energy-sink theory must be an accurate quantitative estimation of this quantity. When one of the two bodies is rigid, Euler's dynamical equations may be applied to the rigid body to determine the torque on it, and hence, in principle, the motor work can be determined. As was pointed out in Ref. 5, Euler's moment equations cannot be applied to the quasirigid body to determine the motor torque since the dampers in the body are also applying torques (unless, of course, the damper torques are known quantities; however, such a situation defeats the purpose of the energy-sink analysis since we have prima facie assumed that these are undeterminable torques as is the case for real spacecraft). Thus, when both the platform and the rotor are quasirigid, the motor torque cannot, and should not, be determined from Euler's equations; an indirect approach and/or additional axiom(s) must be used. Deferring a discussion of the quasirigid platform *and* rotor, we define the core body to be the quasirigid part of the two-body system. Further, the core energy is defined as the rotational kinetic energy of a fictitious rigid body that possesses the inertia properties of the entire gyrostat but moves (in inertial space) exactly like the core body. Obviously, the motion of this fictitious rigid body is *not* governed by torque-free dynamics. Hubert's definition of core energy is more general[13] since it takes into account any potential energy as well; however, in the present analysis, since we are ignoring any flexibility contributions, the two definitions are equivalent.

The kinetic energy of an axially aligned rigid gyrostat can be written as[12]

$$2E = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 + J\Omega^2 + 2J\Omega\omega_3 \tag{13}$$

Despite its rigid-body origins, we will take this expression to represent the "total" kinetic energy of the quasirigid gyrostat. In fact, this is the *fundamental axiom* of quasirigidity and of the energy-sink analysis. Physically, E is actually equal to the measured kinetic energy of a gyrostat, rigid or not, in the sense that the angular rates in Eq. (13) are actual outputs from rate-measuring sensors placed along the principal axes of the bodies. Implicit in the fundamental axiom is the tacit assumption that the dampers do not make a significant contribution to the energy value (but obviously alter it, albeit very little, by dissipating energy). In other words, although the dampers do not significantly alter energy, they make a fundamental contribution to the *rate* of change of energy (and hence the nutational stability). Clearly, the energy-sink analysis should not be applied when the dampers violate this condition. For example, spin-stabilized spacecraft that use liquid propellant for apogee kick do not always satisfy the fundamental axiom (due to the possibility of fuel slosh).

When the platform is the core body, the core energy is defined as

$$2E_C = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \tag{14}$$

Similarly, when the rotor is the core body, the core energy takes the form

$$2E_C' = I_1\omega_1^2 + I_2\omega_2^2 + I_3(\omega_3 + \Omega)^2 \tag{15}$$

Hence, Eq. (13) can be rewritten as

$$2E = 2E_C + J\Omega^2 + 2J\Omega\omega_3 \tag{16}$$

Differentiating the previous equation, we have

$$\dot{E} = \dot{E}_C + J\Omega(\dot{\Omega} + \dot{\omega}_3) + J\dot{\Omega}\omega_3 \tag{17}$$

When the rotor is rigid, the torque applied by the motor to the rotor is also the net torque on the rotor and hence can be

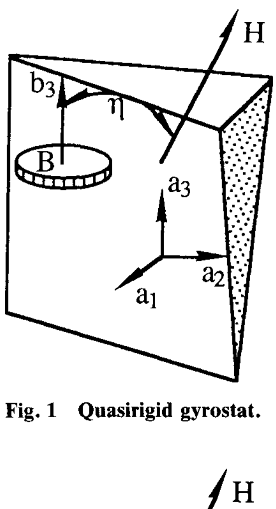

Fig. 1 Quasirigid gyrostat.

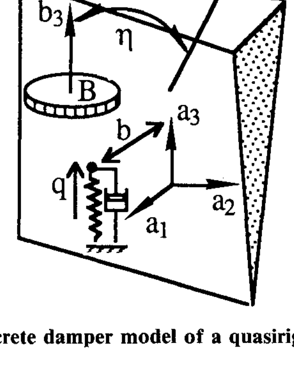

Fig. 2 Discrete damper model of a quasirigid gyrostat.

obtained from Euler's moment equations. Further, since the rotor is axisymmetric,

$$T_{R/M} = T_R = J(\dot{\Omega} + \dot{\omega}_3) \tag{18}$$

Note that $T_{P/M} = -T_{R/M}$ but $T_P \neq -T_R$ since the platform experiences additional torques due to the dampers. The rate of work done by this torque to maintain relative motion is therefore

$$\dot{W} = T_R \Omega = J\Omega(\dot{\Omega} + \dot{\omega}_3) \tag{19}$$

Hence, Eq. (17) may be rewritten as

$$\dot{E} = \dot{E}_C + \dot{W} + J\dot{\Omega}\omega_3 \tag{20}$$

Comparing this with Eq. (12), we obtain the equation for the rate of change of energy dissipation—when the platform is the core body—to be

$$\dot{E}_D = \dot{E}_C + J\dot{\Omega}\omega_3 \tag{21}$$

Clearly, the core energy is not guaranteed to decrease; the necessary and sufficient condition for the dissipation of core energy is

$$\dot{E}_D < J\dot{\Omega}\omega_3 \tag{22}$$

Obviously, when the right-hand side of Eq. (22) is nonnegative (e.g., a rotor spin-up condition), then the core energy decreases; on the other hand, if

$$J\dot{\Omega}\omega_3 < \dot{E}_D \tag{23}$$

then the core energy will increase (e.g., a sufficiently rapid rotor spin-down/despin condition will increase the core energy). Hence, if the nutation damper is designed such that

$$|\dot{E}_D| > |J\dot{\Omega}\omega_3| \tag{24}$$

the core energy is insured to decrease. For the special case of a constant relative rotor spin speed, Eq. (24) is always satisfied and any damper will minimize the core energy. Further, from Eq. (21), it follows that the rate of change of core energy is equal to the rate of change of the damper energy,

$$\dot{E}_D = \dot{E}_C \tag{25}$$

As explained elsewhere, when the rotor is the core body, Eq. (18) cannot be used to determine the motor torque; this must be evaluated by applying Euler's dynamical equations to the rigid platform. Unfortunately, for an asymmetric platform, the torque equation contains the gyroscopic term, and hence no elegant expression for the motor torque can be obtained. To circumvent this problem, if we restrict our attention to a symmetric platform, then there is no difference between the rotor and the platform besides the difference in their respective inertia values; mathematically, Eq. (21) and the resulting conclusions remain unaltered. Hence, for now, we restrict our attention to a core asymmetric platform and a rigid axisymmetric rotor.

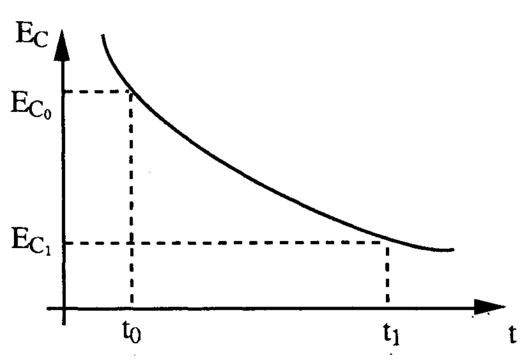

Fig. 3 Postulated core energy dissipation function.

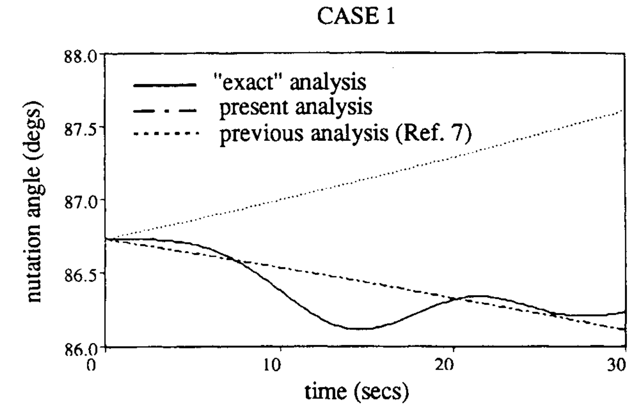

Fig. 4 Quantitative and qualitative agreement of present analysis.

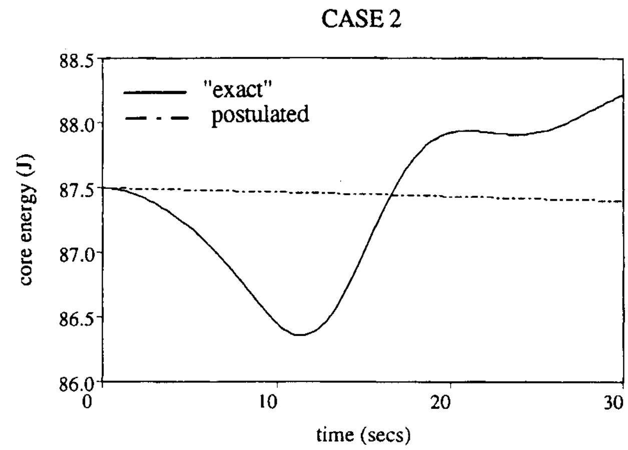

Fig. 5 Core energy time history for a very short period of time.

Nutational Motion

With the understanding of the discussion following Eq. (13), the angular momentum of an axisymmetric quasirigid gyrostat may be written as[12]

$$H^2 = I_t^2(\omega_1^2 + \omega_2^2) + (I_s\omega_3 + J\Omega)^2 \tag{26}$$

(Although it is sometimes assumed in the literature, it is worth pointing out that an axisymmetric gyrostat does not necessarily imply an axisymmetric platform and vice versa. This is because it is quite possible to obtain an inertia ellipsoid of revolution for the entire gyrostat without requiring the same to be true for the platform.[12]) Instead of total energy, we use the core energy to eliminate the transverse angular velocity. Of course, the most general energy to use would be the damper energy, but this is not integrable from Eq. (21). Hence, this section is restricted to the special case of constant relative rotor spin speed. Since

$$2E_C = I_t(\omega_1^2 + \omega_2^2) + I_s\omega_3^2 \tag{27}$$

the spin component of the angular velocity may be solved for in terms of the angular momentum and core energy by eliminating $(\omega_1^2 + \omega_2^2)$ from Eqs. (26) and (27); from the resulting quadratic,

$$I_s(I_s - I_t)\omega_3^2 + 2JI_s\Omega\omega_3 + 2E_C I_t + J^2\Omega^2 - H^2 = 0 \tag{28}$$

we have

$$\omega_3 = \frac{J\Omega}{I_t - I_s} \pm \sqrt{Q}\left(\frac{I_t}{I_t - I_s}\right)\frac{1}{I_s} \tag{29}$$

where

$$Q = 2E_C I_s\left(1 - \frac{I_s}{I_t}\right) - H^2\left(\frac{I_s}{I_t}\right)\left(1 - \frac{I_s}{I_t}\right) + \left(\frac{I_s}{I_t}\right)J^2\Omega^2 \tag{30}$$

As before, at $t = 0$, the sign indeterminacy of Eq. (29) must be determined from consistency with the initial condition on ω_3; at any other point in time, continuity will insure a unique sign. Substituting Eq. (29) into Eq. (7), we arrive at

$$\eta = \cos^{-1}\left[\left(\frac{I_t}{I_t - I_s}\right)\frac{1}{H}\{J\Omega \pm \sqrt{Q}\}\right] \tag{31}$$

The (asymptotic) stability condition may be established by differentiating this equation. Thus,

$$\dot{\eta} = \frac{-\dot{E}_C}{H\sin\eta}\left(\frac{I_s}{\pm\sqrt{Q}}\right) \tag{32}$$

and stability ($\dot{\eta} < 0$) dictates that the negative sign of the radical be chosen (since $\dot{E}_C < 0$), resulting in

$$\eta = \cos^{-1}\left[\left(\frac{I_t}{I_t - I_s}\right)\frac{1}{H}\{J\Omega - \sqrt{Q}\}\right] \tag{33}$$

The choice of this sign when imposed on Eq. (29) implies

$$(I_t - I_s)\omega_3 - J\Omega = -\sqrt{Q}(I_t/I_s) \tag{34}$$

which yields the well-known dual-spin stability condition[3,4]

$$(I_t - I_s)\omega_3 - J\Omega \leq 0 \tag{35}$$

Of course, the elegance of this analysis is that it does not produce any contradiction [to the condition of Eq. (1)]. This condition is sometimes restated as (for $\omega_3 > 0$)

$$\frac{I_s + J\Omega/\omega_3}{I_t} \geq 1 \tag{36}$$

to conform with the major-axis rule with the understanding that the left-hand side of Eq. (36) is equivalent to the "effective inertia ratio."

As a final note, the reader is directed to Appendix A where the nonnegativity of the parameter Q and its associated properties are outlined.

Numerical Simulations

The main purpose of the numerical simulations is to establish the correctness of the present theory by comparing its predictions with those of a discrete damper model (mass-spring-dashpot system: see Fig. 2) that serves as a yardstick for "numerical experimentation." The equations of motion for the discrete damper system are given in Ref. 7 and are provided in Appendix B (consistent with the notation used in this paper). The quantitative prediction of the nutation angle from Eq. (33) can be compared with the "exact" analysis by numerically integrating the equations of motion [Eqs. (B1-B4)] provided the energy dissipation function of the damper is known. Following Ref. 7, we postulate a damper energy dissipation function (which is also the core energy function for the case of constant rotor relative spin) according to (see Fig. 3)

$$E_D(t) = E_C(t) = \frac{E_{C_1}E_{C_0}(t_1 - t_0)}{E_{C_1}t_1 - E_{C_0}t_0 - (E_{C_1} - E_{C_0})t} \tag{37}$$

where E_{C_1} is guessed/predicted and $E_{C_0} = E_C(t_0)$ is known from initial conditions. Before proceeding any further, it is worth mentioning several points.

1) Since the energy sink does not assume any specific nutation damper, the quantitative predictions must be interpreted as mean values.

2) The predictions of Eq. (33) are based on a postulated energy function, and it is hence function specific.

3) The power of the energy-sink analysis is in its qualitative predictions in the large and its quantitative predictions in the small.

For the purpose of comparison, we use the same values of Ref. 7; thus, the numerical simulations are performed for the following inertia values of the gyrostat: $I_s = 100$ kg m^2, $I_t = 175$ kg m^2, $J = 1$ kg m^2, and $m_G = 990$ kg, possessing the damper characteristics given by $m_p = 10$ kg, $b = 1$ m, $k = 10$ N/m, and $c = 2$ N s/m, with initial conditions, $q = \dot{q} = 0$. For all of the cases, the rotor relative spin is a constant 10 rad/s; the cases differ only in the choice of the initial conditions of the platform angular velocity. For case 1,

$$\omega_1 = \omega_3 = 0 \text{ rad/s} \qquad \omega_2 = 1 \text{ rad/s}$$

According to Eq. (35), we should expect stability that concurs with the plot shown in Fig. 4. The initial core energy for this case is 87.5 J, and the plot in Fig. 4 was obtained using a postulated final core energy of 87.4 J. To study this case in a little more detail, a plot of core energy is shown in Fig. 5. At first glance, it appears to contradict the theory, although the behavior of the nutation angle concurs. To recognize the fallacy of this hasty conclusion, Fig. 6 shows the core energy time history for a much longer period of time (1000 s); notice the eventual decrease of the core energy and the agreement (mean value) with the postulated function. It is worthwhile to point out that this behavior is not noticeable until the simulation is carried out for a sufficiently long period of time; as an illustration, the plot for 200 s is shown in Fig. 7. The oscillatory behavior of the "exact" core energy may be attributed to the energy interchange between the damper and the platform since

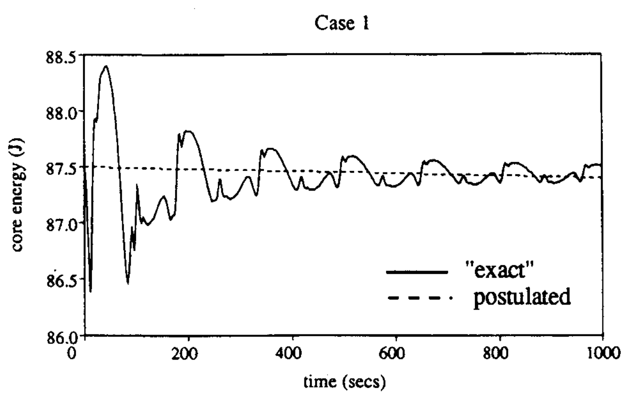

Fig. 6 True behavior of core energy—ultimate dissipation.

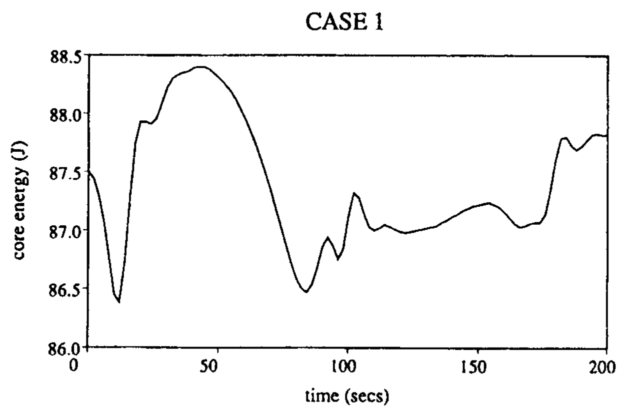

Fig. 7 Core energy time history for a short period of time.

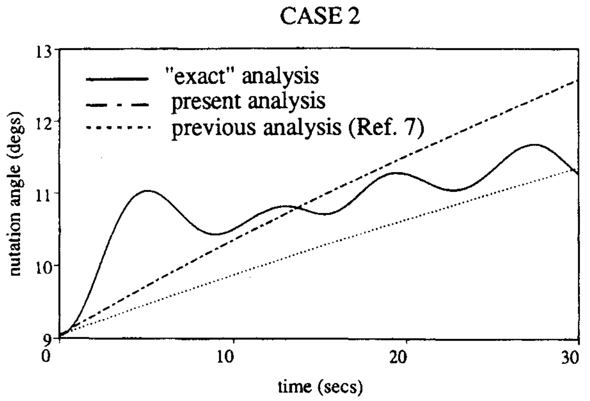

Fig. 8 Agreement in nutation angle prediction for an unstable gyrostat.

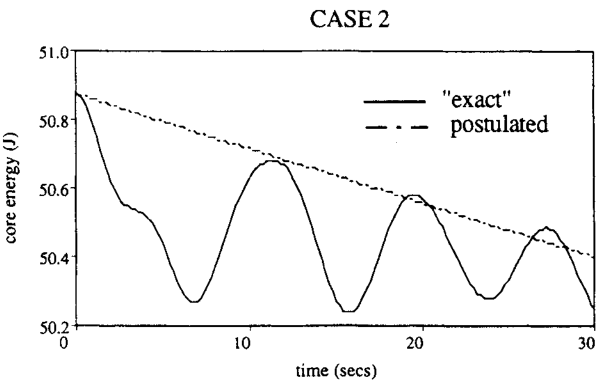

Fig. 9 Core energy dissipation for an unstable gyrostat.

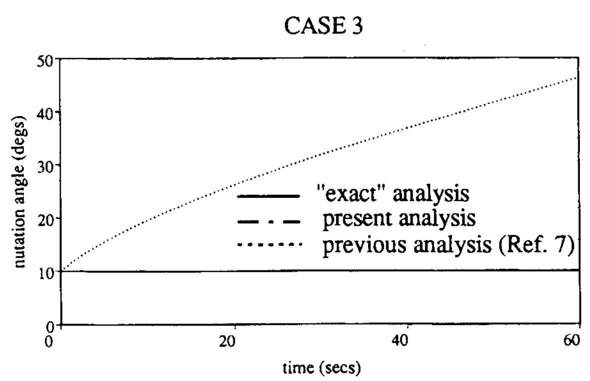

Fig. 10 Near-perfect corroboration of present energy-sink analysis.

the potential energy of the damper mass is not included in the energy-sink formulation.

Case 2 corresponds to the condition $\omega_1 = 0.1$ rad/s, $\omega_2 = 0$ rad/s, and $\omega_3 = 1$ rad/s. In this case, Eq. (35) predicts instability that once again concurs with the plots shown in Fig. 8. The initial core energy for this case is 50.875 J, and the plot was obtained (with the positive sign on the radical) for a postulated final core energy of 50.7 J. Notice that the core energy even for this case decreases (Fig. 9) as it should if the theory is correct. The reason the analysis of Ref. 7 is in agreement here is because the total energy actually decreases for this case, and hence from Eq. (11) the nutation angle grows; to reiterate, from the previous analysis, the nutation angle always grows for total energy dissipation, which is clearly incorrect.

Finally, in Ref. 7, case 3—$\omega_1 = \omega_3 = 0$ rad/s and $\omega_2 = 0.01$ rad/s—is used to show the "failure of [the] energy-sink method for 'non-flat' spin"; in the present analysis, the agreement is actually more pronounced as seen from Fig. 10 where it is almost impossible to distinguish between the exact analysis and the present analysis. Contrary to the conclusions of Ref. 7, this case further strengthens the energy-sink analysis.

Conclusions

There is no doubt that a motor in a dual-spin spacecraft can pump mechanical energy into the system, especially for the special case of a constant relative angular speed of the rotor. However, when it is properly taken into account, the energy-sink analysis yields consistent results in concurrence with Landon's original idea. In fact, regardless of whether the motor is adding *or removing* energy, the work done by the motor torque must be taken into account. In a sense, the term energy sink is a misnomer since it is possible for the total energy to increase; perhaps a better term is "energy-state approximation." Of course, this is simply semantics; the important issue addressed here is the reconciliation of this form of analysis when interpreted in various ways. Since, for systems containing driven rotors, we have no a priori knowledge of the monotonicity of the total (mechanical) energy, it is pointless to mathematically relate it to the nutation angle. When applied correctly, the energy-sink analysis not only yields consistent results but also relationships that assist in the design of the damper.

Appendix A: Parameter Q

Substituting Eq. (26) and (27) into Eq. (30), we have

$$\frac{Q}{I_s}\left(\frac{I_s}{I_t - I_s}\right) = \Omega^2\left(\frac{J^2}{I_t - I_s}\right) + \omega_3^2(I_t - I_s) - 2J\omega_3 \qquad (A1)$$

which reduces to

$$Q = (I_s/I_t)\left[(I_t - I_s)\omega_3 - J\Omega\right]^2 \geq 0 \qquad (A2)$$

Thus Q is nonnegative with the minimum value zero; for $Q = 0$,

$$\omega_3 = \frac{J\Omega}{I_t - I_s} = (\omega_3)_{\max} \qquad (A3)$$

where the maximum condition follows from Eq. (29) (assuming a stable prolate gyrostat). The nonnegativity of Q also establishes a condition on the core energy

$$Q \geq 0 \Rightarrow E_C \geq \frac{1}{2}\left[\frac{H^2}{I_t} - \frac{(J\Omega)^2}{I_t(1 - I_s/I_t)}\right] \qquad (A4)$$

Hence, the core energy attains its minimum concurrent with that of Q. We now show that these values are identical to the condition of zero nutation angle that we call the condition at infinity (technically, $t \to \infty$). Hence, if

$$\eta = 0 \qquad (A5)$$

represents the condition at infinity, then the transverse components of the angular velocity vanish, which implies that

$$(\omega_3)_\infty = (\omega_3)_{\max} \qquad (A6)$$

Hence, from Eqs. (A2) and (A3) we must necessarily have

$$Q_\infty = 0 \qquad (A7)$$

Finally, from Eqs. (A5) and (7) we have

$$H_\infty = I_s\omega_3 + J\Omega = \frac{J\Omega}{1 - I_s/I_t} \qquad (A8)$$

where the last equality follows from Eqs. (A6) and (A3); substituting this expression into Eq. (A4), we arrive at

$$E_C \geq \tfrac{1}{2}(\omega_3)_\infty^2 I_s \qquad (A9)$$

which shows that the minimum value of the core energy is attained at infinity.

Appendix B: Equations of Motion for the Discrete Damper System

The following equations of motion are given in Ref. 7 and are reproduced here to conform with the present notation:

$$(I_t + \mu q^2)\dot{\omega}_1 - \mu b q \dot{\omega}_3 - \mu q(\omega_2\omega_3 q + \omega_1\omega_2 b - 2\omega_1 \dot{q})$$
$$- (I_t - I_s)\omega_2\omega_3 + J\Omega\omega_2 = 0 \tag{B1}$$

$$(I_t + \mu q^2)\dot{\omega}_2 - \mu b \ddot{q} + \mu\left[(\omega_1^2 - \omega_3^2)qb + \omega_1\omega_3 q^2 + 2\omega_2 q \dot{q}\right]$$
$$+ (I_t - I_s)\omega_1\omega_3 - J\Omega\omega_1 = 0 \tag{B2}$$

$$I_s\dot{\omega}_3 - \mu b q \omega_1 + \mu b(\omega_2\omega_3 q - 2\omega_1\dot{q}) = 0 \tag{B3}$$

$$\mu b \dot{\omega}_2 - \mu \ddot{q} + \mu\left[q(\omega_1^2 + \omega_2^2) - \omega_1\omega_3 b\right] - kq - c\dot{q} = 0 \tag{B4}$$

where

$$\mu = \frac{m_p m_G}{m_p + m_G} \tag{B5}$$

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